

INTEGRAL REPRESENTATION OF CONTINUOUS FUNCTIONS⁽¹⁾

BY

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I. INTRODUCTION

It was shown by F. Riesz [5; 350]⁽²⁾ that every subharmonic function u can be represented as the sum of the potential of its mass distribution plus a harmonic function; the potential appears in the form of a Stieltjes integral (Riesz's theorem is stated in (2.2.1)). We prove that the Stieltjes integral may be replaced by a Lebesgue integral if u is continuous, and if the lower generalized Laplacian of u is less than $+\infty$, except possibly on a set of capacity zero (Theorem II). In other words, the above assumptions imply the absolute continuity of the mass distribution associated with u .

We obtain this result as a consequence of Theorem I, which deals with the representation of continuous functions in integral form. In another paper, Theorem I will be used in an investigation of the uniqueness theory for Laplace series. The theorem was actually suggested by this application, and is of a type similar to a theorem of Zygmund [9; 276] on the representation of continuous functions of one variable.

Our results are stated for the plane, but analogous theorems evidently hold for continuous functions of three or more variables, if the generalized Laplacians are suitably defined (see, for instance, [7]).

1.1. *Notation.* Let D be a finite plane domain (that is, a connected open set not containing the point at infinity). Let Z be a closed and bounded plane set of capacity zero (see 2.3). Let $J(P, r)$ denote the closed circular disc bounded by the circle $C(P, r)$ with center at P , and radius r . If the function $F \in L$ on $C(P, r)$, we put

$$(1.1.1) \quad \Delta_r F(P) = \frac{1}{2\pi r} \int_{C(P, r)} F(Q) ds_Q - F(P).$$

If there exists $r_0 > 0$ such that $F \in L$ on $C(P, r)$ for $0 < r < r_0$, we define the generalized Laplacian of F at P by

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⁽²⁾ Numbers in brackets refer to the bibliography at the end of the paper.

$$(1.1.2) \quad \Delta F(P) = \lim_{r \rightarrow 0} 4\Delta_r F(P)/r^2,$$

provided the limit exists. The upper and lower Laplacians, $\Lambda^*F(P)$ and $\Lambda_*F(P)$, are defined likewise, with \limsup and \liminf in place of \lim .

By Wiener's sequence method [8; 25] it is possible to define Green's function for any bounded domain R . We therefore do not have to restrict ourselves to Dirichlet domains. We shall introduce some more notation in 3.1.

1.2. THEOREM I. *Let the function F be continuous in D . Suppose*

(i) $\Lambda^*F(P) > -\infty$, $\Lambda_*F(P) < +\infty$ for P in $D - D \cdot Z$;

(ii) *there exists a function y , defined in D , such that $y \in L$ on every closed subset of D , and such that $y(P) \leq \Lambda^*F(P)$ for P in D .*

Then

(a) $\Delta F(P)$ *is finite at almost all points P of D ;*

(b) $\Delta F \in L$ *on every closed subset of D ;*

(c) *at almost all points P of D we have*

$$(1.2.1) \quad \int \int_{J(P,r)} |\Delta F(Q) \log PQ| dQ < +\infty \quad (J(P,r) \subset D);$$

(d) *at all points P of D at which (1.2.1) holds, and for all bounded domains R which contain P and whose closure is contained in D , we have*

$$(1.2.2) \quad F(P) = -\frac{1}{2\pi} \int \int_R \Delta F(Q) g(P, Q) dQ + H(P),$$

where $g(P, Q)$ is Green's function for R , and H is harmonic in R and assumes the values of F on the boundary of R (in the sense of Wiener [8; 25], if R is not a Dirichlet domain).

1.3. THEOREM II. *Let the function u be continuous and subharmonic in D . Suppose that $\Lambda_*u(P) < +\infty$ for P in $D - D \cdot Z$. Then, at all points P of D , and for all bounded domains R which contain P and whose closure is contained in D , we have*

$$(1.3.1) \quad u(P) = -\frac{1}{2\pi} \int \int_R \Lambda u(Q) g(P, Q) dQ + H(P),$$

where $g(P, Q)$ is Green's function for R , and H is the least harmonic majorant of u in R . In addition, if D is bounded, and if u has a harmonic majorant in D , then (1.3.1) holds if $R = D$.

Evidently, (1.2.2) (and (1.3.1)) may also be written in the form

$$(1.3.2) \quad F(P) = \frac{1}{2\pi} \int \int_R \Delta F(Q) \log PQ dQ + h(P),$$

in which the integral represents the potential of a mass distribution of density ΔF , and h is harmonic in R .

II. SUBHARMONIC FUNCTIONS, SETS OF CAPACITY ZERO

2.1. DEFINITION. A function u is said to be subharmonic in D if [4; 333]

- (a) u is upper semi-continuous in D ,
- (b) $u(P) < +\infty$ in D ,
- (c) $u(P)$ is finite on a dense set in D ,
- (d) for every domain G with boundary B , such that $G+B \subset D$, the inequality $u(P) \leq H(P)$ on B implies $u(P) \leq H(P)$ in G for every function H which is harmonic in G and continuous on $G+B$.

The condition (d) may be replaced by

$$(2.1.1) \quad \Delta_r u(P) \geq 0$$

provided $J(P, r) \subset D$ [4; 334], or by

$$(2.1.2) \quad \Lambda^* u(P) \geq 0$$

for P in D [3; 14].

2.2. LEMMA. If u is subharmonic in D , then $\Lambda u(P)$ is finite at almost all points P of D , and $\in L$ on every closed subset of D .

Proof. There exists a non-negative set function μ , called the mass distribution of u , such that (1) $\mu(e)$ is finite for any Borel set e contained with its closure in D , (2) if G is any open set contained with its boundary in D , then μ is absolutely additive on G , and

$$(2.2.1) \quad u(P) = \int \int_G \log PQ d\mu_Q + h(P) \quad (P \text{ in } G),$$

where h is harmonic in G [5; 350], [7; 452].

Defining the symmetric derivative [6; 149] by

$$(2.2.2) \quad D_{\text{sym}} \mu(P) = \lim_{r \rightarrow 0} \frac{1}{\pi r^2} \mu(J(P, r)),$$

we have, by the two-dimensional case of a theorem of Saks [7],

$$(2.2.3) \quad \Lambda u(P) = 2\pi D_{\text{sym}} \mu(P)$$

at all points P at which $D_{\text{sym}} \mu(P)$ is finite.

Since μ is additive, the derivative $D\mu(P)$ (and, a fortiori, $D_{\text{sym}} \mu(P)$) is finite p. p. and $\in L$ on E , where E is any measurable set for which $\mu(E)$ is finite. Combining these results, we obtain the lemma.

2.3. *Sets of capacity zero.* Let E be a closed and bounded plane set whose complement contains an unbounded domain U , and which is contained in the interior I of a circle C . Let Γ be the boundary of U . Put $V = U \cdot I$. Then

V is a domain. Let u be the function which is harmonic in V and which is the sequence solution [8; 25] of the Dirichlet problem, corresponding to the boundary values 1 on Γ and 0 on C .

The set E is said to be of capacity zero if the function u thus determined vanishes in V [1; 208]. This definition is independent of the choice of C .

Closed and bounded plane sets of capacity zero, which we shall denote by the letter Z , have the following geometric properties [1]:

(2.3.1) If C is a circle, then $Z \cdot C$ is of measure zero on C .

(2.3.2) If G is a domain, then $G - G \cdot Z$ is a domain.

We shall also make use of the following result:

(2.3.3) Let G be a domain; Let $Z \subset G$; let u be subharmonic in $G - Z$ and bounded above in a neighborhood of Z . If we define $u(P)$ on Z by

$$u(P) = \limsup_{Q \rightarrow P, Q \notin Z} u(Q),$$

then the extended function is subharmonic in G [2; 31].

The following lemma will enable us to infer that the function H in (1.2.2) and (1.3.1) is harmonic.

2.4. LEMMA. *Let E be a set of measure zero, contained in the finite domain R . Let F be defined on $R - E$. Suppose $F \in L$ on $J(P, r)$ if $J(P, r) \subset R$, and suppose $\Delta_r F(P) = 0$ for P on $R - E$, if $J(P, r) \subset R$. Then there exists a function H which is harmonic in R , such that $H(P) = F(P)$ on $R - E$.*

Proof. It is easily verified that the function

$$H(P) = H_r(P) = \frac{1}{\pi r^2} \iint_{J(P, r)} F(Q) dQ \quad (P \in R; J(P, r) \subset R)$$

is independent of r for fixed P on $R - E$, coincides with $F(P)$ on $R - E$, is continuous in R , hence independent of r for fixed P on E . It follows that H has the Gaussian mean value property for all $J(P, r) \subset R$.

III. THE OPERATOR Ω

It will be convenient to have a short symbol for integrals of the type appearing in (1.2.2) and (1.3.1). Hence we make the following definition.

3.1. DEFINITION. Let $f \in L$ on R , where R is a bounded domain. Put $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$. We define

$$(3.1.1) \quad \Omega f(P) = \Omega_R f(P) = -\frac{1}{2\pi} \iint_R f(Q) g(P, Q) dQ$$

at all points P of R at which at least one of the integrals

$$\iint_R f^+(Q) g(P, Q) dQ, \quad \iint_R f^-(Q) g(P, Q) dQ$$

is finite. $g(P, Q)$, as before, is Green's function for R . It will not be necessary in the sequel to indicate the dependence of Ω on R .

Let $J(P, r) \subset R$, and suppose $f \in L$ on R . We shall use the following properties of Ω :

3.2. Ωf^+ and Ωf^- are subharmonic in R .

3.3. $\Omega f \in L$ on $C(P, r)$, and $\Omega f \in L$ on $J(P, r)$; hence $\Omega f(P)$ is finite at almost all points P of R .

3.4. If f is bounded above in R , then Ωf is lower semi-continuous and $> -\infty$ in R .

3.5. If P is a point at which $\Omega f(P)$ is defined, then

$$(3.5.1) \quad \Delta_r \Omega f(P) = \frac{1}{2\pi} \iint_{J(P, r)} f(Q) \log \frac{r}{PQ} dQ,$$

$$(3.5.2) \quad \lim_{r \rightarrow 0} \frac{1}{2\pi r} \int_{C(P, r)} \Omega f(Q) ds_Q = \Omega f(P).$$

3.6. At almost all points P of R , $\Delta \Omega f(P) = f(P)$.

3.7. If f is upper semi-continuous and less than $+\infty$ at P , then $\Delta^* \Omega f(P) \leq f(P)$.

3.8. If f is bounded above in R , and if Z is a closed set of capacity zero, then, for P in R ,

$$\liminf_{Q \rightarrow P, Q \notin Z} \Omega f(Q) = \Omega f(P).$$

3.9. **Proofs of 3.2–3.8.** We can write $g(P, Q) = h(P, Q) - \log PQ$, where $h(P, Q)$ is harmonic in P , for each Q , and is symmetric in P, Q . Then

$$(3.9.1) \quad \Omega f^+(P) = -\frac{1}{2\pi} \iint_R f^+(Q) h(P, Q) dQ + \frac{1}{2\pi} \iint_R f^+(Q) \log PQ dQ.$$

Let these two integrals be denoted by $\phi_1(P)$ and $\phi_2(P)$ respectively. The function ϕ_1 is clearly continuous in R . Since

$$\Delta_r h(P, Q) = \frac{1}{2\pi r} \int_{C(P, r)} h(M, Q) ds_M - h(P, Q) = 0,$$

we have, by Fubini's theorem on the change of order of integration,

$$\Delta_r \phi_1(P) = -\frac{1}{2\pi} \iint_R f^+(Q) \Delta_r h(P, Q) dQ = 0,$$

which shows that ϕ_1 is harmonic in R . Next, ϕ_2 is subharmonic in R ; this follows from [5; 328], with $\mu(e) = \iint_e f^+(Q) dQ$. Thus 3.2 is proved.

By 3.2, Ωf^+ and $\Omega f^- \in L$ on $C(P, r)$ and on $J(P, r)$ [3; 4]. 3.3 follows.

If f is bounded above in R , $\Omega f^+(P)$ is finite at all P of R . Hence $\Omega f(P)$

$> -\infty$ in R . Since Ωm is continuous in R for any constant m , we may assume without loss of generality that $f(P) \leq 0$ in R . Lower semi-continuity (and hence 3.4) follows from 3.2 and 2.1.

Next, (3.5.1) is a consequence of Fubini's theorem and the formula

$$\Delta_r \log PQ = \frac{1}{2\pi r} \int_{C(P,r)} \log MQ ds_M - \log PQ = \begin{cases} 0 & (PQ > r), \\ \log r/PQ & (PQ \leq r). \end{cases}$$

By 3.2, (3.5.2) holds for f^+ and for f^- [3; 8], and hence for f .

Noting that $\lim_{r \rightarrow 0} (1/\pi r^2) \iint_{J(P,r)} f(Q) dQ = f(P)$ p. p. in R [6; 118], we see that 3.2, (3.9.1), and (2.2.3) imply $\Lambda \Omega f^+(P) = f^+(P)$ p. p. in R . 3.6 follows.

To prove 3.7, we note that f is bounded above in a neighborhood of P . Hence $\Omega f(P)$ is defined. Suppose first that $f(P)$ is finite. Let $\epsilon > 0$ be arbitrary. There exists $\delta > 0$ such that $f(Q) \leq f(P) + \epsilon$ for Q in $J(P, \delta)$. By (3.5.1) we have, for $0 < r < \delta$,

$$\Delta_r \Omega f(P) \leq \frac{1}{2\pi} \iint_{J(P,r)} (f(P) + \epsilon) \log \frac{r}{PQ} dQ = \frac{1}{4} r^2 (f(P) + \epsilon).$$

Hence $\Lambda^* \Omega f(P) \leq f(P) + \epsilon$, which implies $\Lambda^* \Omega f(P) \leq f(P)$. Next, suppose $f(P) = -\infty$. Let $m > 0$ be arbitrary. There exists $\delta > 0$ such that $f(Q) \leq -m$ for Q in $J(P, \delta)$. Proceeding as above, we see that $\Lambda^* \Omega f(P) \leq -m$, which implies $\Lambda^* \Omega f(P) = -\infty$.

To prove 3.8, we note that Ωf is defined in R , and that

$$\liminf_{Q \rightarrow P, Q \notin Z} \Omega f(Q) \geq \liminf_{Q \rightarrow P} \Omega f(Q) \geq \Omega f(P) \quad (P \text{ in } R)$$

(by 3.4). Thus, if 3.8 is false, there exists a number m such that

$$\liminf_{Q \rightarrow P, Q \notin Z} \Omega f(Q) > m > \Omega f(P).$$

For Q sufficiently near to P , $Q \notin Z$, we have then $\Omega f(Q) \geq m$. By (2.3.1), Z is of measure zero on every circle; hence

$$\liminf_{r \rightarrow 0} \frac{1}{2\pi r} \int_{C(P,r)} \Omega f(Q) ds_Q \geq m > \Omega f(P),$$

contrary to (3.5.2).

IV. PROOF OF MAIN RESULTS

In the proof of Theorem I we shall make use of the following theorem.

4.1. VITALI-CARATHÉODORY THEOREM. *Let f be defined in R , $f \in L$ on R . Then there exist functions $u_1, u_2, \dots, v_1, v_2, \dots$, defined in R , with the following properties:*

(a) $u_1(P) \leq u_2(P) \leq \dots \leq f(P) \leq \dots \leq v_2(P) \leq v_1(P)$ in R ;

(b) each u_n is bounded above and is upper semi-continuous in R , each v_n is bounded below and is lower semi-continuous in R ;

(c) $u_n \in L$, $v_n \in L$ on R , $n = 1, 2, 3, \dots$;

(d) $u_n(P) \rightarrow f(P)$, $v_n(P) \rightarrow g(P)$ as $n \rightarrow \infty$, at almost all points P of R .

For the proof we refer to [6; 75].

4.2. Proof of Theorem I. Let u be one of the upper semi-continuous functions associated with y in R , in the sense of the Vitali-Carathéodory Theorem. Put

$$(4.2.1) \quad W(P) = F(P) - \Omega u(P) \quad (P \text{ in } R).$$

By 3.4, W is upper semi-continuous and $< +\infty$ in R . Applying 3.7 to u , we have, for P in $R - R \cdot Z$,

$$(4.2.2) \quad \begin{aligned} \Lambda^* W(P) &\geq \Lambda^* F(P) - \Lambda^* \Omega u(P) \geq \Lambda^* F(P) - u(P) \\ &\geq \Lambda^* F(P) - y(P) \geq 0. \end{aligned}$$

Here we used assumption (ii) and the first half of assumption (i) of Theorem I.

By 3.2, $W(P)$ is finite on a dense set in the domain $R - R \cdot Z$ (see (2.3.2)). Thus, by (2.1.2), W is subharmonic in $R - R \cdot Z$.

Let $M \in R \cdot Z$. By (2.3.2), there exists a simple closed curve Γ in R whose interior G contains M , such that $\Gamma \cdot Z = 0$. Being upper semi-continuous and less than $+\infty$ in R , W is bounded above in G . Moreover, by 3.8,

$$(4.2.3) \quad \begin{aligned} \limsup_{Q \rightarrow M, Q \notin Z} W(Q) &= \lim_{Q \rightarrow M} F(Q) - \liminf_{Q \rightarrow M, Q \notin Z} \Omega u(Q) \\ &= F(M) - \Omega u(M) = W(M). \end{aligned}$$

Since $G \cdot Z$ is closed, we can now apply (2.3.3), and conclude that W is subharmonic in G , and thus in R . By 2.2, $\Lambda W(P)$ is finite p. p. in R . By 3.6,

$$(4.2.4) \quad \Lambda F(P) = \Lambda W(P) + \Lambda \Omega u(P) = \Lambda W(P) + u(P) \quad (\text{p. p. in } R).$$

Since R is an arbitrary bounded domain in D , it follows that $\Lambda F(P)$ is finite p. p. in D . Similarly, 2.2 shows that $\Lambda F \in L$ on any closed subset of R , and hence on any closed subset of D . By 3.3, (1.2.1) holds p. p. in D . We have thus proved (a), (b), and (c) of Theorem I.

Next, put

$$(4.2.5) \quad f(P) = (\Lambda^* F(P) + \Lambda_* F(P))/2$$

at all points P of D at which the sum in (4.2.5) is defined, and put $f(P) = 0$ at all other points of D . Then

$$(4.2.6) \quad \Lambda_* F(P) \leq f(P) \leq \Lambda^* F(P) \quad (P \text{ in } D),$$

$$(4.2.7) \quad f(P) = \Lambda F(P) \quad (\text{p. p. in } D).$$

Let $u_1, u_2, \dots, v_1, v_2, \dots$, be functions associated with f and R in the

sense of 4.1. Let E be the set on which both $\Omega u_1(P)$ and $\Omega v_1(P)$ are finite. It follows, by (a) of 4.1, that $\Omega u_n(P)$, $\Omega v_n(P)$ ($n=1, 2, 3, \dots$), as well as $\Omega f(P)$, are finite for P on E . By 3.3, E contains almost all points of R . Let $J(M, r) \subset R$, with M in E . This choice of M assures that all integrals used in the remainder of this proof are finite. Put

$$(4.2.8) \quad W_n(P) = F(P) - \Omega u_n(P) \quad (P \in R; n = 1, 2, 3, \dots).$$

Proceeding as in (4.2.2) and (4.2.3), with u_n and f in place of u and y , we see that W_n is subharmonic in R . By (2.1.1)

$$(4.2.9) \quad \Delta_r W_n(M) \geq 0 \quad (n = 1, 2, 3, \dots).$$

By (3.5.1), (4.2.8), and (4.2.9),

$$(4.2.10) \quad \Delta_r F(M) \geq \frac{1}{2\pi} \iint_{J(M, r)} u_n(Q) \log \frac{r}{PQ} dQ \quad (n = 1, 2, 3, \dots).$$

The sequence $\{u_n(P)\}$ converges to $f(P)$ monotonically p. p. in R . Hence we may pass to the limit in (4.2.10), and obtain

$$(4.2.11) \quad \Delta_r F(M) \geq \frac{1}{2\pi} \iint_{J(M, r)} f(Q) \log \frac{r}{PQ} dQ.$$

In the same manner, working with the functions $v_n(P)$, we obtain (4.2.11) with the inequality reversed. Hence the equality sign holds in (4.2.11), and comparison with (3.5.1) shows that

$$(4.2.12) \quad \Delta_r(F(M) - \Omega f(M)) = 0.$$

By 3.3, the function $F - \Omega f$ thus satisfies all hypotheses of 2.4. Hence there exists a function H , harmonic in R , such that

$$(4.2.13) \quad F(P) = \Omega f(P) + H(P) \quad (\text{p. p. in } R).$$

Now let N be a point in R at which (1.2.1) holds. By (4.2.7), $\Omega f(N)$ is finite. Since F and H are continuous in R , we have

$$(4.2.14) \quad \begin{aligned} F(N) - H(N) &= \lim_{r \rightarrow 0} \frac{1}{\pi r^2} \iint_{J(N, r)} (F(Q) - H(Q)) dQ \\ &= \lim_{r \rightarrow 0} \frac{1}{\pi r^2} \iint_{J(N, r)} \Omega f(Q) dQ = \Omega f(N), \end{aligned}$$

the last equality being a consequence of (3.5.2).

4.3. We have now proved Theorem I, except for the assertion concerning the boundary values of H .

Put $u_1(P) = \Omega f^+(P)$, $u_2(P) = \Omega f^-(P)$, for P in R . By 3.2, u_1 and u_2 are subharmonic in R ; the least harmonic majorants of u_1 and u_2 in R are both zero [5; 357].

Fix a point M in R . Choose a sequence $\{R_n\}$ of domains with polygonal boundaries B_n , such that $M \in R_1$, $R_1 \subset R_2 \subset \cdots \subset R$, $R_n \rightarrow R$ as $n \rightarrow \infty$. The functions u_1 and $u_2 \in L$ on B_n , since they are subharmonic [2; 14]. Since $\Omega f = u_1 - u_2$, this shows that Ωf exists p. p. on B_n . We have shown that (4.2.13) holds wherever Ωf is finite. Hence (4.2.13) holds at almost all points of B_n , which implies

$$(4.3.1) \quad H_n(M) = h_n^{(1)}(M) - h_n^{(2)}(M) + H(M),$$

where H_n is harmonic in R_n and coincides with F on B_n , and $h_n^{(1)}, h_n^{(2)}$, are the best harmonic majorants of u_1, u_2 , in R_n .

(For the concepts of best harmonic majorants (B.H.M.) and least harmonic majorants (L.H.M.) of a subharmonic function in a given domain we refer to [4; 334], [5; 357–358]. Radó [3; 39] proved that the L.H.M. is equal to the B.H.M. for Dirichlet domains. Since the B.H.M. is defined for Dirichlet domains only, it would seem that Radó's theorem has made the concept of B.H.M. superfluous.)

Since $\lim h_n^{(1)}(M) = 0 = \lim h_n^{(2)}(M)$ as $n \rightarrow \infty$ [5; 358], (4.3.1) implies that $\lim H_n(M) = H(M)$. This completes the proof of Theorem I.

4.4. Proof of Theorem II. All hypotheses of Theorem I are evidently satisfied (we can take $y=0$). To show that (1.3.1) holds everywhere, it is sufficient, by (d) of Theorem I, to show that $\Omega f(P) > -\infty$ for P in R , where $f = \Lambda * u \geq 0$. By 3.4, Ωf is upper semi-continuous in R , hence the equation $\Omega f(P) = -\infty$ would imply $\lim \Omega f(Q) = -\infty$ as $Q \rightarrow P$, and Ωf would not be equal to the continuous function $u - H$ almost everywhere in R .

The remaining assertions of Theorem II were proved by Riesz [5; 357] for the Stieltjes integral representation. Hence they evidently hold in our case as well.

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